Combinatorica **27** (3) (2007) 253–267
DOI: 10.1007/s00493-007-2086-y

COMBINATORICA
Bolyai Society – Springer-Verlag

INDEPENDENT SYSTEMS OF REPRESENTATIVES IN WEIGHTED GRAPHS

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Received June 15. 2003
Revised January 3, 2005

The following conjecture may have never been explicitly stated, but seems to have been floating around: If the vertex set of a graph with maximal degree Δ is partitioned into sets V_i of size 2Δ, then there exists a coloring of the graph by 2Δ colors, where each color class meets each V_i at precisely one vertex. We shall name it the *strong* 2Δ*-colorability conjecture*. We prove a fractional version of this conjecture. For this purpose, we prove a weighted generalization of a theorem of Haxell, on independent systems of representatives (ISR's). En route, we give a survey of some recent developments in the theory of ISR's.

1. Independent systems of representatives

Consider a graph G and a partition $V_1,V_2,\dots,V_m$ of its vertex set. A choice of one vertex from each set V_i is called an *independent system of representatives* (ISR) if the selected vertices are non-adjacent in G. An ISR for a partial family of $V_1,V_2,\dots,V_m$ is called a *partial ISR*. When wishing to stress that an ISR is not properly partial, that is, it is for the entire family $V_i, i\le m$, we sometimes call it a *total* ISR. The requirement that the sets V_i are disjoint is made here only for sake of convenience. The general case, in which the sets may intersect, can be reduced to the case of disjoint sets

Mathematics Subject Classification (2000): 05C15

* The research of the first author was supported by grant no 780/04 from the Israel Science Foundation, and grants from the M. & M. L. Bank Mathematics Research Fund and the fund for the promotion of research at the Technion.

† The research of the third author was supported by the Sacta–Rashi Foundation.

by the following device. Given a vertex x, form a copy of x for each i such that $x \in V_i$, and put this copy in V_i. In the resulting graph connect all copies of x to each other. The newly formed sets V_i are disjoint, and an ISR for them in the resulting graph corresponds in a natural way to an ISR in the original graph.

Independent systems of representatives arise in various contexts, for example in hypergraph matchings and in graph colorings and list colorings. Algorithmically, the problem of determining whether a family of sets has an ISR is NP-complete. This is shown, for example, by the fact that the problem of determining whether a 3-uniform 3-partite hypergraph with equal parts has a perfect matching can be reduced to it. Thus a good characterisation for the existence of ISR's cannot be expected. But recently some results have been found providing sufficient conditions. One of these, due to Haxell, is that if all V_i's are of size at least $2\Delta(G)$, then there exists an ISR (see Corollary 3.2 below). But there are quite a few indications that something much stronger is true: under this condition, there exists a decomposition of $V(G)$ into $\max_{1\le i\le m}|V_i|$ partial ISR's. This conjecture does not appear in the literature, but it seems that authors, like Alon and Haxell, who have studied partition-compatible colorings of graphs as above (these are called "strong colorings"), have suspected it. We shall call it the *strong 2Δ-colorability conjecture*. One aim of this paper is to prove the fractional version of this conjecture, namely, that under the above condition the characteristic function of the vertex set of the graph is the sum of the characteristic functions of partial ISR's, each multiplied by some coefficient, and the sum of these coefficients is $\max_{1\le i\le m}|V_i|$.

In order to prove this, we shall need a result relating the maximal total weight of an ISR in a graph with given weights on its vertices, to a weighted domination parameter that is defined below.

But first, let us describe some general results on ISR's.

2. Topology and domination

An interesting recent development in the theory of ISR's is the application of topological methods. The basic result in this direction is formulated in terms of the topological connectivity of the complex of independent sets in the graph.

A non-empty collection $\mathcal{C}$ of sets is called a *simplicial complex* if it is hereditary, namely if $\sigma \in \mathcal{C}$ and $\tau \subseteq \sigma$ imply $\tau \in \mathcal{C}$. It is well known (see e.g. [6]) that every simplicial complex has a unique (up to homeomorphism) geometric realisation, namely an embedding in some space R^n, in which every simplex $\sigma \in \mathcal{C}$ is realized as a homeomorph of a simplex in R^n. We shall identify a complex with its geometric realisation. The *topological connectivity* of a simplicial complex $\mathcal{C}$ is the largest number c such that for every number $k \leq c$, every embedding of the k-dimensional sphere S^k is extendable to an embedding of the $k+1$-dimensional ball B^{k+1} in $\mathcal{C}$. The connectivity of a complex may be infinite. The connectivity of $\mathcal{C}$, plus 2, is denoted by $\eta(\mathcal{C})$ (the reason for this definition is that the addition of 2 makes the formulation of some results more elegant).

For a graph H, we denote by $\mathcal{I}(H)$ the simplicial complex consisting of all independent sets of vertices in H.

As before, let V_i, $i \leq m$, be a partition of the vertex set of a graph G. Given a vertex $v \in V(G)$, we write $i(v)$ for the index i for which $v \in V_i$. For a set Z of vertices we write $I(Z) = \{i(z) : z \in Z\}$. As usual, we denote the set $\{1, \ldots, m\}$ by $[m]$. Given a subset I of $[m]$, we write V_I for $\bigcup_{i \in I} V_i$.

The link between topological connectivity and ISR's was established in [5], in which the following is implicit (see also [15], where it was first explicitly formulated):

Theorem 2.1. *If for all $I \subseteq [m]$*

$$\eta(\mathcal{I}(G[V_I])) \geq |I|$$

then the partition $(V_i : i \leq m)$ of $V(G)$ has an ISR.

To exemplify these notions and the above theorem, consider a bipartite graph G, whose sides are the two parts V_1 and V_2 of the given partition. In this case there exists an ISR if and only if the graph is not complete bipartite. But not being complete bipartite means the existence of a connection in $\mathcal{I}(G)$ between the two simplices V_1 and V_2 of $\mathcal{I}(G)$. Thus, not being complete bipartite is tantamount to $\mathcal{I}(G)$ being connected, which, in the above terminology, means being 0-connected, which means that $\eta(\mathcal{I}(G)) \geq 2$. Thus, in this example, the condition of the theorem is not only sufficient, but also necessary.

In order to apply Theorem 2.1 in combinatorial settings, it suffices to prove results of the type "$\eta \geq \iota$", where ι is some combinatorially defined parameter of graphs. This would mean that in the theorem one can replace η by ι. Indeed, such results have been found. In all of them, ι is some kind of domination number. Here are some definitions pertaining to domination in graphs.

For a vertex x in a graph we write $\tilde{N}(x)$ for the punctured neighborhood of x, namely the set of vertices adjacent to x. We write $N(x) = \tilde{N}(x) \cup \{x\}$. For a set X of vertices we write $N(X) = \bigcup_{x \in X} N(x)$ and $\tilde{N}(X) = \bigcup_{x \in X} \tilde{N}(x)$.

Definition 2.2. Let H be a graph and X a subset of $V(H)$. A subset D of $V(H)$ is called X-*dominating* if $X \subseteq N(D)$. A $V(H)$-dominating set is simply called *dominating*. A set D is called *totally dominating* if $V(H)=\tilde{N}(D)$.

The following are some graph parameters measuring how hard it is to dominate the graph:

- The *domination number*, $\gamma(H)$, of H is the smallest size of a dominating set.
- The *total domination number*, $\tilde{\gamma}(H)$, of H is the minimal size of a totally dominating set.
- The *edge domination number*, $\gamma^E(H)$, of H, is the minimal number of edges whose union (namely, the set of all vertices appearing in them) is a dominating set.
- The *independent domination number*, $i\gamma(H)$, of H is the maximum, over all independent sets I in H, of the minimal size of a set dominating I. The *bi-independent domination number*, $i\gamma i$, of H is the maximum, over all independent sets I in H, of the minimal size of an independent set dominating I.
- A function $f: V(H) \to \mathcal{R}^+$ is called *fractionally dominating* (or simply *dominating* if it is clear from the context that the dominating object in question is a function) if $\Sigma_{u\in N(v)} f(u) \geq 1$ for every vertex v. It is called *weakly dominating* if $\Sigma_{u\in \tilde{N}(v)} f(u) + deg(v)f(v) \geq 1$ for every vertex v. The minimal sum of the values of a dominating function in H is denoted by $\gamma^*(H)$. The minimal sum of values of a weakly dominating function is denoted by $\gamma^*_w(H)$.

Here are some results relating domination parameters and the connectivity parameter η:

- $\eta(\mathcal{I}(G)) \geq i\gamma(G)$, in fact even $\eta \geq i\gamma i$ ([5]).
- $\eta(\mathcal{I}(G)) \geq \tilde{\gamma}(G)/2$ ([15], see also [3]. The combinatorial corollary was proved earlier, by Haxell [11], as is described in the next section).
- $\eta(\mathcal{I}(G)) \geq \gamma^E(G)$ (implicit in [15] and [3]. Also follows directly from the main result of [16]).
- $\eta(\mathcal{I}(G)) \geq \gamma^*_w(G)$ [16].

In [16] there was proved a theorem from which all the above results follow directly. Here is a formulation of it, not explicitly given in the original paper, but easily derivable from the proof of the main result:

Theorem 2.3. *Let ψ be a function from the class of graphs to the set of positive integers together with ∞. Suppose that ψ satisfies the following properties:*
(i) $\psi(K_0)=0$,
(ii) For every graph G there exists an edge $e=(x,y)$ of G such that

$$\psi(G-e) \geq \psi(G)$$

(where $G-e$ is obtained from G by the removal of e), and

$$\psi(G-N(\{x,y\})) \geq \psi(G)-1.$$

Then $\eta(\mathcal{I}(G)) \geq \psi(G)$.

In fact, there is a maximal function ψ_0 satisfying the conditions of the theorem. It is best described in terms of a game between two players, (I) and (II). Player (I) wants to maximize the function ψ in the theorem (and hence prove that η is large), while player (II) wants to minimize ψ. Player (I) selects an edge $e=(x,y)$ in the graph given at the present stage of the game. Player (II) chooses between two possibilities: he either (1) deletes e from the graph, or else (2) deletes all neighbors of x and y (including, of course, x and y themselves). The game ends when either there remains an isolated vertex, in which case ψ is defined as ∞, or there are no remaining vertices, in which case ψ is defined as the number of moves of Player (II) of type 2.

We define $\psi_0(G)$ as the maximal value of $\psi(G)$ Player (I) can achieve in the game. Theorem 2.3 then states that $\eta(\mathcal{I}(G)) \geq \psi_0(G)$. We suggest the following conjecture:

Conjecture 2.4.

$$\eta(\mathcal{I}(G)) = \psi_0(G).$$

In general, γ cannot be bounded from above in terms of $i\gamma$. For example, the line graph of the hypergraph $\binom{[n^2]}{n}$ satisfies $i\gamma=1$, $\gamma=n$ (this example is due to Roy Meshulam [17]). But there are interesting classes of graphs for which $i\gamma=\gamma$. We shall call a graph *stably wide* if this property holds for each of its induced subgraphs.

In [2] it was shown that chordal graphs are stably wide. Another interesting class of stably wide graphs is that of cycles of length $3k$ (in which $i\gamma=\gamma=k$. It is possible to show that for such graphs also $\eta(\mathcal{I}(G))=k$).

3. Haxell's theorem

As already noted, each of the results in the list in the previous section, combined with Theorem 2.1, yields a combinatorially formulated sufficient condition for the existence of ISR's. The combination of the second result in the list with the theorem yields that if $\tilde{\gamma}(G[V_I])/2 \geq |I|$ for every $I \subseteq [m]$ then there exists an ISR. In fact, this was proved before Theorem 2.1, in a combinatorial way, in [11]. Let us re-state it a bit differently:

Theorem 3.1. *If $\tilde{\gamma}(G[V_I]) \geq 2|I|-1$ for all $I \subseteq [m]$, then there exists an ISR.*

In [11] a special case of this theorem is proved, but the proof actually yields this result. We denote by $\Delta(G)$ the maximal degree of a vertex in G.

Corollary 3.2. *Any partition V_i, $i \leq m$ of $V(G)$ into sets of size at least $2\Delta(G)$ has an ISR. In fact, given any vertex v, there exists an ISR containing v.*

Proof. Consider the system of sets $(V_j : j \neq i(v))$, together with the singleton $\{v\}$. Our aim is to show that this system has an ISR in the graph induced on its union by G. Every k sets in this system contain together at least $2\Delta(k-1)+1$ vertices, and hence need at least $(2\Delta(k-1)+1)/\Delta > 2(k-1)$ vertices to totally dominate them. Thus the condition of the theorem holds, proving the existence of an ISR. ∎

It is known that the corollary is sharp: in [18] an example is given of a graph G and a partition of G into $2\Delta(G)$ sets, each of size $2\Delta(G)-1$, having no ISR.

Theorem 2.1, together with the fact cited above that $\eta \geq i\gamma$, yield the following analogous corollary:

Corollary 3.3. *If G is stably wide, then any partition V_i, $i \leq m$ of $V(G)$ into sets of size at least $\Delta(G)+1$ has an ISR. In fact, every vertex belongs to some such ISR.*

(The number is $\Delta+1$, rather than Δ, since $i\gamma$ is equal to γ, not to $\tilde{\gamma}$.)

Corollary 3.3 implies, for example, a conjecture of Du, Hsu and Hwang (see [7]), that in any partition of a cycle of length $3k$ into k sets of size 3 there exists an ISR. In [8] this conjecture was proved in the stronger form of strong colorability. But the corollary yields the following stronger result:

Corollary 3.4. *If G is the disjoint union of cycles of length divisible by 3, then in any partition into sets of size 3 there exists an ISR.*

For further applications of Theorem 3.1, we shall need a refinement of it. Namely, we need more precise information on the dominating set, in case that an ISR does not exist.

Theorem 3.5. *If there does not exist an ISR, then for some $I \subseteq [m]$ there exists $D \subseteq V_I$, of size at most $2|I|-2$, that totally dominates $G[V_I]$. Furthermore, D can be written as $X \cup Y$, where*
(i) $|Y| < |I|$,
(ii) The vertices of Y form a (strictly) partial ISR for the family $(V_i, i \in I)$, and
(iii) The subgraph $G[D]$ induced on D is the union of stars, covering the entire set D, whose centers belong to X and the ray-vertices belong to Y (in particular, $|X| \le |Y|$).

All this is implicitly proved in [11]. But since this is not quite transparent from the proof there, we provide here a full proof.

Proof. Let $J \subset [m]$ be a set of maximal cardinality such that $(V_j : j \in J)$ has an ISR. By assumption, $J \neq [m]$. Choose any $i \in [m] \setminus J$ and any vertex $x_1 \in V_i$. Among all ISR's of $(V_j : j \in J)$ choose one, say R_1, such that x_1 has a minimal number of neighbors in R_1. Let Y_1 be $\tilde{N}(x_1) \cap R_1$. Clearly, Y_1 is non-empty, or else $R_1 \cup \{x_1\}$ would be an ISR for $J \cup \{i\}$, contradicting the maximality property of J.

If $D_1 = \{x_1\} \cup Y_1$ is totally dominating in $G_1 = G[V_{I(D_1)}]$ then the theorem is proved, with $I = J \cup \{i\}$, $D = D_1$, $X = \{x_1\}$ and $Y = Y_1$. Thus we may assume that there exists a vertex x_2 of G_1 not dominated by D_1. Since $G[D_1]$ does not contain isolated vertices, $x_2 \notin D_1$.

Among all ISR's of J agreeing with R_1 on $I(D_1) \setminus \{i\}$, pick one, say R_2, such that x_2 has a minimal number of neighbors in R_2. Denote by Y_2 the set of neighbors of x_2 in R_2. Suppose that $Y_2 = \emptyset$. If $x_2 \in V_i$ then $J \cup \{i\}$ has an ISR, contradicting the maximality of J. If $x_2 \in V_j$ for some $j \in I(R_1)$ then replacing the representative of V_j in R_1 by x_2 results in an ISR in which x_1 has fewer neighbors than in R_1, contradicting the minimality property of R_1. Thus Y_2 can be assumed to be non-empty.

Let $D_2 = \{x_1, x_2\} \cup Y_1 \cup Y_2$, and let $G_2 = G[V_{I(D_2)}]$. If D_2 is totally dominating in G_2, then the theorem is proved, with $I = \{i\} \cup \{j \in I(D_2)\}$, $D = D_2$ and $X = \{x_1, x_2\}$, $Y = Y_1 \cup Y_2$.

Thus we may assume that there exists a vertex x_3 of G_2 not dominated by D_2. The argument now goes on as before. Since our setting is finite (for example, m is finite), this process must terminate at some point, yielding the theorem. ∎

Remark. The topological proofs of the theorem do not yield Theorem 3.5 in full. But in [4] a topological proof is given for a substantial part of it: the missing point is that Y forms a partial transversal. It is proved there only that Y can be assumed to be independent.

4. Weighted domination

To the setting of a graph G with a partition $V_1, V_2, \ldots, V_m$ we now add another element: a real valued weight function w on V. We write ν_w for the maximal possible total weight of an ISR, namely $\nu_w = \max \sum_{x \in P} w(x)$, where the maximum is taken over all partial ISR's P. The origin of this notation is in the notation for the maximal total weight of a matching in a weighted hypergraph. Indeed, our results generalize the theory of weighted matchings in hypergraphs, in the same way that ordinary ISR theory generalizes matching theory for hypergraphs. Since we allow partial ISR's, we may ignore all vertices with non-positive weight w, and hence we may assume that all weights $w(x)$ are positive.

Given any real-valued function h on a set S, and a subset T of S, we write $h[T]$ for $\sum_{t \in T} h(t)$. We also write $|h|$ for $h[S]$. A pair of non-negative functions $g\colon [m] \to \mathbb{R}^+$ and $f\colon V \to \mathbb{R}^+$ is said to be *w-dominating* if for every vertex x we have

$$g(i(x)) + f[\tilde{N}(x)] \geq w(x).$$

For such a pair of functions (not necessarily dominating) we write

$$|(g,f)| = |g| + |f|/2.$$

We write τ_w for $\min\{|(g,f)|\colon (g,f) \text{ is dominating}\}$.

Our main theorem is:

Theorem 4.1. $\tau_w \leq \nu_w$.

This is a generalization of the weighted version of König's theorem. Understanding why this is so can clarify the definition of τ_w, and in particular the factor $\frac{1}{2}$ on $|f|$ appearing in it.

The setting of the weighted version of König's theorem is that of a bipartite graph with sides A, B and a weight function w on its edges. A pair of functions $g : A \to \mathbb{R}^+$, $g : B \to \mathbb{R}^+$ is said to be *dominating* if $g(a) + f(b) \geq w((a,b))$ for every edge (a,b). The theorem then says that the maximal sum of weights of edges in a matching is equal to the minimum of $|g| + |f|$ over all dominating pairs (g,f).

This setting can be transformed into ours, as follows. Assign a vertex $x(e)$ to each edge e of the graph, and a set V_i to each vertex v_i in the first side of the graph. Put $x(e)$ in V_i if e is adjacent to v_i. Finally, form a graph G on the set of vertices $x(e)$, $e \in E$ by joining $x(e)$ to $x(e')$ if e, e' meet in the second side of the graph. Thus every vertex $b \in B$ corresponds in G to a clique C_b. But in our setting a vertex does not dominate itself, and hence in order to dominate C_b, instead of a value $f(b)$ assigned to b by the dominating function in König's theorem, we need to assign the value $f(b)$ to *two* vertices in C_b. To return to the original weighted version of König's theorem, a factor $1/2$ in the measuring of the weight of f is needed.

In fact, we shall need a strengthened version of Theorem 4.1. For every $1 \le i \le m$ let $t_i = \max\{w(y) : y \in V_i\}$.

Theorem 4.2. $\tau_w \le \nu_w$. *Furthermore, it is possible to find a dominating pair of functions* (g, f) *of weight at most* ν_w *such that*
(*) $f(x) + g(i(x)) \le t_{i(x)}$ *for every vertex* x.
If all weights $w(v)$ *are integral, then* g *and* f *can also be assumed to be integral.*

Before proving the theorem, let us explain why it is a generalization of Theorem 3.1. In the setting of the latter, $w \equiv 1$. Assume that Theorem 4.2 is true, and consider the case that there is no ISR, namely $\nu_w < m$. Then, by Theorem 4.2, there exists an integral valued dominating pair of functions (g, f) such that $|(g, f)| < m$. Clearly, it is possible to assume that both g and f are $0,1$ functions. Let $I = \{i : g(i) = 0\}$. Then the support of f totally dominates $G[V_I]$. By Condition (*) in Theorem 4.2, the support of f is contained in V_I. The fact that $|(g, f)| \le m - 1$ means that $|f| \le 2(|I| - 1)$, namely $\tilde{\gamma}(G_I) \le 2(|I| - 1)$, which is the desired conclusion in Theorem 3.1.

The reverse inequality in Theorem 4.1 is generally false. In fact, there is no upper bound for ν_w in terms of τ_w. To see this, for any positive integer k let G be the complete bipartite graph $G = K_{k,k}$, let $m = 2k$ and let $(V_1, \ldots, V_{2k})$ be a partition of $V(G)$ into singletons. Let $w \equiv 1$. Then $\tau_w = 1$ and $\nu_w = k$.

Proof of Theorem 4.2. By an approximation argument we may assume that the weights are rational. Multiplying by the common denominator of the weights, we can then assume that all weights w are integral. The proof will be by induction on $w[V]$. In the case $w[V] = 0$ there is nothing to prove. Let now s be an integer, such that the theorem is true for $w[V] < s$. We shall prove the theorem in the case that $w[V] = s$.

For each $i \le m$ let $M_i = \{x \in V_i : w(x) = t_i\}$, namely M_i is the set of vertices of maximal weight in V_i. If the sets M_i have an ISR, then $\nu_w = \sum t_i$, while the pair of functions defined by $f\equiv 0$, $g(i)=t_i$ is dominating. Clearly, (*) is satisfied in this case, and since $|(g,f)|=\sum t_i$, the theorem is proved. Thus we may assume that the sets M_i do not have an ISR in the subgraph H of G induced on their union. By Theorem 3.5, there exists then a non-empty subset I of $[m]$, and a totally dominating set D in $H[M_I]$ of the form $D=X\cup Y$, where Y is a properly partial ISR for the sets M_i, $i\in I$, and the graph induced by G on D is a union of stars, having centers in X and ray-vertices in Y.

Define a weight function w' by $w'(v)=w(v)-|\tilde{N}(v)\cap D|$. Remove all vertices with non-positive w'. If (g',f') is a dominating pair for w', then $(g',f'+\chi_D)$ is a dominating pair for w, where χ_D is the characteristic function of D. Clearly, also, if (g',f') satisfies (*) for w', then $(g',f'+\chi_D)$ satisfies (*) for w. By the induction hypothesis there exists a dominating pair (g',f') for w' satisfying (*) and $|(g',f')|\leq \nu_{w'}$. Since $|(g',f'+\chi_D)|=|(g',f')|+\frac{|D|}{2}$, in order to complete the proof it suffices to show that $\nu_w\geq \nu_{w'}+|D|/2$. Since $|D|\leq 2|Y|$, it suffices to show $\nu_w\geq\nu_{w'}+|Y|$.

Choose an ISR T with $w'[T]=\nu_{w'}$, having the additional property that its intersection with Y is maximal. We claim that every vertex $y\in Y\setminus T$ is connected to some vertex in T. Assume that this is not the case for some $y\in Y\setminus T$. By the domination property of D, every remaining vertex x satisfies $w'(x)\leq t_{i(x)}-1$. Since $w'(y)=t_{i(y)}-1$, taking y instead of the representative in T of $i(y)$ yields an ISR T' with $w'(T')=w'[T]$ and $|T'\cap Y|>|T\cap Y|$, contradicting the maximality property of T.

By the definition of w' it follows that $w'[T]\leq w[T]-|Y|$, implying $\nu_w\geq \nu_{w'}+|Y|$, as required. ∎

We conjecture that a similar weighted extension holds for the result of [5]. That result states that if $i\gamma(G[V_I])\geq |I|$ for all $I\subseteq [m]$ then there exists an ISR. The weighted version should be:

Conjecture 4.3. Let $(V_1,\dots,V_m)$ be a partition of the vertex set of a graph G, and let a nonnegative weight $w(v)$ be given for every vertex v of the graph. Then there exists a system of weights $g(i)$, $1\leq i\leq m$, such that for every independent set I there exists a nonnegative real valued function f on $V(G)$, satisfying the conditions:
(1) $g(i(v))+f[\tilde{N}(v)]\geq w(v)$ for every $v\in I$, and
(2) $|g|+|f|\leq \nu_w$.

If true, this conjecture would imply, in a way similar to the proof in [1], a weighted version of Ryser's conjecture for 3-partite 3-graphs, as follows: assume that to every edge e of a 3-partite 3-graph H there is assigned a non-negative real weight $w(e)$. Then $\tau_w(H)\leq 2\nu_w(H)$, where in this case

$\tau_w(H)$ is defined as $\min\{\sum f(v) : \sum_{v\in e} f(v) \geq w(e)$ for every $e \in E(H)\}$ and $\nu_w(H)$ is the maximal sum of weights of a matching in H.

5. Strong colorability and strong choosability

Definition 5.1. Let $V_1, V_2, \dots, V_m$ be a partition of the vertex set of a graph G. A proper coloring of G is said to *respect* the partition if each color meets each V_i at most once.

In other words, a coloring respects the partition if it is a proper coloring of the graph obtained from the original graph by adding to it cliques on all sets V_i.

Definition 5.2. A graph G is said to be *strongly k-colorable* if for every partition of its vertex set into parts of size at most k, there is a coloring of G by k colors, respecting the partition. A graph is called *strongly k-choosable* if for every partition of its vertex set into parts of size at most k, and each assignment $L(v)$ of a list of colors of size k to each vertex v, there is a coloring c of G respecting the partition, such that $c(v) \in L(v)$ for every vertex v.

It is known [9] that the disjoint union of cliques of size k is strongly k-choosable (or, equivalently: the line graph of a bipartite graph Γ is $\Delta(\Gamma)$-choosable). It is also known [8] that a cycle of length divisible by 3 is strongly 3-choosable.

In [13] it was proved that every graph is strongly $3\Delta-1$-colorable, where, as usual, Δ denotes the maximal degree of a vertex in the graph. Here is a short proof of a slightly weaker result. It is based on an idea from the original proof, but it uses Theorem 3.1 directly, rather than going into its technique of proof as is done in [13]. Our proof also yields a stronger result for stably wide (for example, chordal) graphs.

Theorem 5.3. *(i) Every graph is strongly 3Δ-colorable. (ii) Every stably wide graph is strongly $2\Delta+1$-colorable.*

Proof. Let V_i, $i \leq m$ be a partition of $V(G)$ into sets of size at most 3Δ. By adding isolated vertices, if necessary, we can assume that in fact $|V_i| = 3\Delta$ for all i. Let c be a maximal partial proper coloring of G with 3Δ colors, respecting the partition. Assuming the negation hypothesis, there exists a vertex v not colored by c. Without loss of generality, $v \in V_1$, and thus there exists a color, say 1, not appearing in V_1. From each set V_i containing a vertex (say u_i) of color 1, remove all vertices whose color appears among the neighbors of u_i. Let V_i', $i \leq m$ be the resulting sets ($V_i' = V_i$ if V_i does not contain a vertex of color 1). Since $|V_i|\geq 3\Delta$, we have $|V_i'|\geq 2\Delta$, and hence by Corollary 3.2 the sets V_i' have an ISR T containing v. Color the vertices of T by color 1. For each i such that V_i contains a vertex u_i colored 1 by c, color u_i by $c(t)$, where $t\in T\cap V_i'$. Since $t\in V_i'$, no vertex adjacent to u_i is colored by $c(t)$, and hence this coloring is proper. The number of vertices colored by the resulting coloring is greater by at least one than the number of vertices colored by c, contradicting the maximality property of c.

The proof of Part (ii) of the theorem follows the same outline, the difference being the use of Corollary 3.3 instead of Corollary 3.2. ∎

In [14] Haxell showed that the term 3Δ in Theorem 5.3 can be replaced by $(3-\epsilon)\Delta$ for some small number ϵ. By the remark following Corollary 3.2 it is impossible to replace the number 3Δ by a number smaller than 2Δ. But the following is plausible:

Conjecture 5.4. Every graph G is strongly $2\Delta(G)$-colorable.

Possibly, something even stronger is true:

Conjecture 5.5. Every graph G is strongly $2\Delta(G)$-choosable.

In the spirit of the special role played by stably wide graphs in Theorem 5.3, it is tempting to conjecture that for such graphs the conjecture can be improved to $\Delta+1$-choosability. But this is false, as is witnessed by the following example of Stiebitz: let G be the disjoint union of a 6-cycle and a triangle. Then G is stably wide, but not strongly 3-colorable. (To see that G is not strongly 3-colorable, take the partition of $V(G)$ into 3 sets of size 3, each consisting of one vertex from the triangle and two antipodal vertices from the 6-cycle.) However, we propose the following:

Conjecture 5.6. A chordal graph G is strongly $\Delta(G)+1$ choosable.

For $m=2$ (that is, for partitions of the graph into two sets of size $\Delta+1$) it is not hard to show that the conjecture is true even for graphs with no induced cycles of length 4.

A supporting piece of evidence for the possible validity of the extension of Conjecture 5.4 from colorability to choosability is that it is true for $m=2$, namely when the graph is partitioned into two sets of size at most 2Δ. The strong colorability, namely the 2Δ-colorability of the graph obtained by adding cliques on each of the two sets, follows then easily from Hall's theorem. The 2Δ-choosability follows from the following theorem (which we suspect to be known but is perhaps not written up anywhere):

Theorem 5.7. *The list-chromatic number of the complement of a bipartite graph is equal to its chromatic number.*

Proof. Let G be the complement of a bipartite graph on n vertices. Since G is perfect, its chromatic number k is equal to its clique number. This, in turn, is equal to the independence number of the complement $\overline{G}$ of G, which is $n-\tau(\overline{G})$, which by König's theorem is equal to $n-\nu(\overline{G})$.

Let a list $L(v)$ of $n-\nu(\overline{G})$ permissible colors be given for every vertex of G. Let M be a matching of size $\nu(\overline{G})$ in $\overline{G}$. Assume, first, that there exists an edge $(x,y)\in M$ such that $L(x)$ and $L(y)$ share a color c. Define $L'(z)=L(z)\setminus\{c\}$ for all $z\in V(G)$, and delete x and y from the graph. The resulting graph G' satisfies $\nu(\overline{G'}) = \nu(\overline{G})-1$, and hence by an inductive hypothesis it has a proper coloring from the lists $L'(z)$. Coloring then both x and y by c completes this to a proper coloring of G from the lists $L(z)$.

Assume next that for every $(x,y)\in M$ we have $L(x)\cap L(y)=\emptyset$. We claim then that it is possible to select a system of distinct representatives from the lists $L(z)$, which forms then the desired list coloring. For this purpose, we need to check Hall's condition, namely that

$$\Big|\bigcup_{z\in Z} L(z)\Big| \geq |Z|$$

for every set Z of vertices. If Z contains a pair x,y of vertices adjacent in M, then $|\bigcup_{z\in Z} L(z)|\geq 2(n-\nu(\overline{G}))\geq n\geq |Z|$. If Z does not contain such a pair, then $Z\leq n-\nu(\overline{G})=|L(z)|$ (where z is any vertex of Z) and hence the desired inequality is satisfied also in this case. ∎

We shall prove here a fractional version of Conjecture 5.4. Fractional k-choosability, when formulated properly, can be shown to be equivalent to fractional k-colorability. Hence, in fact, we shall show fractional strong $2\Delta(G)$-colorability of any graph G.

We say that a graph G is *fractionally strongly k-colorable* if, for every partition of $V(G)$ into sets V_i of size at most k, the fractional coloring number of the graph obtained by adding cliques on all V_i is at most k.

Theorem 5.8. *Every graph is fractionally strongly 2Δ-colorable.*

Proof. Let V_i, $i\leq m$ be a partition of $V(G)$ into sets of size at most 2Δ. As before, we may assume that in fact $|V_i| = 2\Delta$ for all i. Let $\mathcal{T}$ be the set of all partial ISR's of this system. What we have to prove is that there exists a function $c\colon \mathcal{T}\to\mathbb{R}^+$ such that $|c|\leq 2\Delta$ and $\sum_{x\in T} c(T)\geq 1$ for every vertex x. Since each V_i is of size 2Δ, and every vertex in each V_i is met by every partial ISR in at most one vertex, this implies that in fact c is positive only on total (rather than partial) ISR's, and that every vertex is covered by c precisely once. In other words, c is a perfect fractional matching for $\mathcal{T}$.

By the above, the theorem is equivalent to the claim that $\nu^*(\mathcal{T})=2\Delta$. Suppose that $\nu^*(\mathcal{T})<2\Delta$. By the duality theorem of linear programming, we have $\tau^*(\mathcal{T})<2\Delta$. That is, there exists a weight function $h:V(G)\to\mathbb{R}^+$, with $|h|<2\Delta$, such that $h[T]\geq 1$ for every total ISR T.

Let $w(v)=1-h(v)$ for every vertex v. Then $w[T]\leq m-1$ for every ISR T. By Theorem 4.2 there exists therefore a w-dominating pair (g,f) of functions, where $g\colon[m]\to\mathbb{R}^+$, $f\colon V\to\mathbb{R}^+$, and $|(g,f)|\leq m-1$. Recall that the domination property of (g,f) means that $w(x)\leq g(i(x))+f[\tilde{N}(x)]$ for every vertex x. Summing these inequalities over all vertices x, and remembering that every $i\leq m$ appears 2Δ times as $i(x)$ and each vertex appears in at most Δ neighborhoods $\tilde{N}(x)$, yields that

$$|w|\leq 2\Delta|g|+\Delta|f|=2\Delta|(g,f)|\leq 2\Delta(m-1).$$

But since $|h|<2\Delta$ we have $|w|=\sum_{v\in V}(1-h(v))=|V|-|h|=2\Delta m-|h|>2\Delta(m-1)$, a contradiction. ∎

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